Intro Video: Section 3.2
Product Rule and Quotient Rule

Math F251X: Calculus I

Recall: $\frac{d}{dx}\left(f(x)+g(x)\right) = \frac{d}{dx}\left(f(x)\right) + \frac{d}{dx}\left(g(x)\right)$

and $\frac{d}{dx}\left(af(x)+bg(x)\right) = a\frac{d}{dx}\left(f(x)\right) + b\frac{d}{dx}\left(g(x)\right)$

LINEAR OPERATOR?

What can we say about $\frac{d}{dx}\left(f(x)\cdot g(x)\right)$ or $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)$?

Compare $\frac{d}{dx}\left(\frac{x^3}{x^2}\right)$ and $\frac{\frac{d}{dx}(x^3)}{\frac{d}{dx}(x^2)}$

① $\frac{d}{dx}\left(\frac{x^3}{x^2}\right) = \frac{d}{dx}(x) = 1$

$\neq$

② $\frac{\frac{d}{dx}(x^3)}{\frac{d}{dx}(x^2)} = \frac{3x^2}{2x} = \frac{3x}{2}$

Compare $\frac{d}{dx}(x^3)$ and $\frac{d}{dx}(x^2)\cdot\frac{d}{dx}(x)$

① $\frac{d}{dx}(x^3) = 3x^2$ $\neq$ ② $\frac{d}{dx}(x^2)\frac{d}{dx}(x) = (2x)(1) = 2x$

WARNING!!

$$\frac{d}{dx}\left(f(x)\,g(x)\right) \neq \left(\frac{df}{dx}\right)\left(\frac{dg}{dx}\right)$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) \neq \frac{\frac{d}{dx}(f(x))}{\frac{d}{dx}(g(x))}$$

PRODUCT RULE

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = f(x)\cdot\frac{d}{dx}\left(g(x)\right) + g(x)\cdot\frac{d}{dx}\left(f(x)\right).$$

$$(fg)' = f\cdot g' + g\cdot f'$$

The first times the derivative of the second, plus the second times the derivative of the first.

Example: $h(x) = x^2e^x$. What is $h'(x)$?

$$h'(x) = \frac{d}{dx}\left(x^2e^x\right) = x^2\frac{d}{dx}\left(e^x\right) + e^x\frac{d}{dx}\left(x^2\right)$$

$$= x^2e^x + e^x(2x)$$

Just for fun: proof of product rule

Let $q(x) = f(x)g(x)$.

$$\frac{d}{dx}(q(x)) = \lim_{h \to 0} \frac{q(x+h) - q(x)}{h} = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\boxed{f(x+h)g(x+h)} + \boxed{f(x+h)g(x)} - \boxed{f(x+h)g(x)}\ \boxed{-f(x)g(x)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[f(x+h)\big(g(x+h) - g(x)\big) + g(x)\big(f(x+h) - f(x)\big)\right]$$

$$= \lim_{h \to 0} f(x+h)\left(\frac{g(x+h) - g(x)}{h}\right) + \lim_{h \to 0} g(x)\left(\frac{f(x+h) - f(x)}{h}\right)$$

$$= \lim_{h \to 0} f(x+h) \lim_{h \to 0}\left(\frac{g(x+h) - g(x)}{h}\right) + \lim_{h \to 0} g(x) \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$

$$= f(x)\,g'(x) + g(x)\,f'(x)$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$$

Low D-Hi, minus high D-low, Square the bottom, and off we go!

Example: $j(x) = \dfrac{e^x}{x^2+1}$

$$j'(x) = \frac{(x^2+1)\frac{d}{dx}(e^x) - e^x\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= \frac{(x^2+1)e^x - e^x(2x+0)}{(x^2+1)^2}$$